FAST FLUID EXTENSIONS FOR IMAGE REGISTRATION ALGORITHMS

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ABSTRACT

The present paper shows a sytematic way to derive fluid-like registration equations. The novel technique is demonstrated for the case of optical flow-based and diffusion-based registration.

Index Terms— Biomedical image processing, Image registration, Fluid registration, Diffusion registration, Optical flow registration, Nonlinear differential equations, Partial differential equations

1. INTRODUCTION

Nonlinear image registration has various applications in image processing, ranging from three-dimensional reconstruction of serial sections over segmentation with help of image atlases to the measurement of changes during the development of a disease.

The goal of image registration is to find a vector transformation $\vec{u}(\vec{x})$ so that the sample image $S(\vec{x})$ under the transformation $S(\vec{x} - \vec{u}(\vec{x}))$ matches the template image $T(\vec{x})$. Additionally, such transformation $\vec{u}(\vec{x})$ is required to satisfy some smoothness condition. One way to obtain such a transformation is to use physically motivated equations like the Navier-Lamé equation for the deformation $\vec{u}$ [1] or solving the Navier-Lamé equation for the velocity field $\dot{\vec{u}}$ and to use the time integral of the velocity as the displacement field [2]. The variant that uses the two fields $\dot{\vec{u}}(\vec{x})$ and $\vec{u}(\vec{x})$ is called fluid registration and uses the equation

$$\mu\,\Delta\dot{\vec{u}}(\vec{x}) + (\mu + \lambda)\nabla\left(\nabla \cdot \dot{\vec{u}}(\vec{x})\right) = \vec{F}(\vec{x}, \vec{u}). \qquad (1)$$

This is formally the same equation as for the elastic registration, but using the velocity field $\dot{\vec{u}}$ instead of the displacement field $\vec{u}$. Since the Navier-Lamé equation is physically motivated by the elasticy theory it is not easy to extend the equation to include additional constraints, like the supression of vortices [3] or other properties of the displacement field or to other differential operators for the smoothing of the displacement field.

Another possibility to derive a differential equation for the displacement field is to apply a variational approach [4, 5, 3] that combines various conditions for the displacement field to a functional that is mimimized by the solution. This approach typically yields a static, time and velocity independent equation for the displacement field $\vec{u}(\vec{x})$. To obtain a smooth convergence of the nonlinear equation one introduces an artificial time and solves a diffusion-like equation [6, 7]. Since the displacement field is modified by a diffusion like process, the convergence of the variational equations is rather slow comparing to the fluid registration. Due to the nonlinearity of this static equations it is typically hard to avoid local minima and sophisticated multi-resolution techniques have been developed [3].

The present paper offers a way to construct dynamic equations for the velocity field on the basis of variational equations without the help of fluid dynamics.

2. EQUATION OF MOTION

2.1. Undamped Equation of Motion

For the registration of the sample image $S(\vec{x})$ onto the template image $T(\vec{x})$ with $\vec{x} \in \mathbb{R}^2$, a displacement field $\vec{u}(\vec{x}, t) = (u_1(\vec{x}, t), u_2(\vec{x}, t))^T$ with $\left\{u : \mathbb{R}^2 \mapsto \mathbb{R}^2\right\}$ must be found so that

$$\mathcal{V}[\vec{u}] = \frac{1}{2}\int_\Omega \left(T(\vec{x}) - S(\vec{x} - \vec{u}(\vec{x}), t)\right)^2\, d^2\vec{x} \qquad (2)$$

should be minimal. The dispacement field should obey some smoothness condition. The fluid extension applies this smoothness constraint not to the time derivative of the displacement field

$$\frac{d\vec{u}}{dt}(\vec{x}, t) = \dot{\vec{u}}(\vec{x}, t),$$

and we will use the function $\mathcal{T}[\dot{\vec{u}}]$ for the smoothness of the Lagrangian. For the case of optical flow-based image regis-

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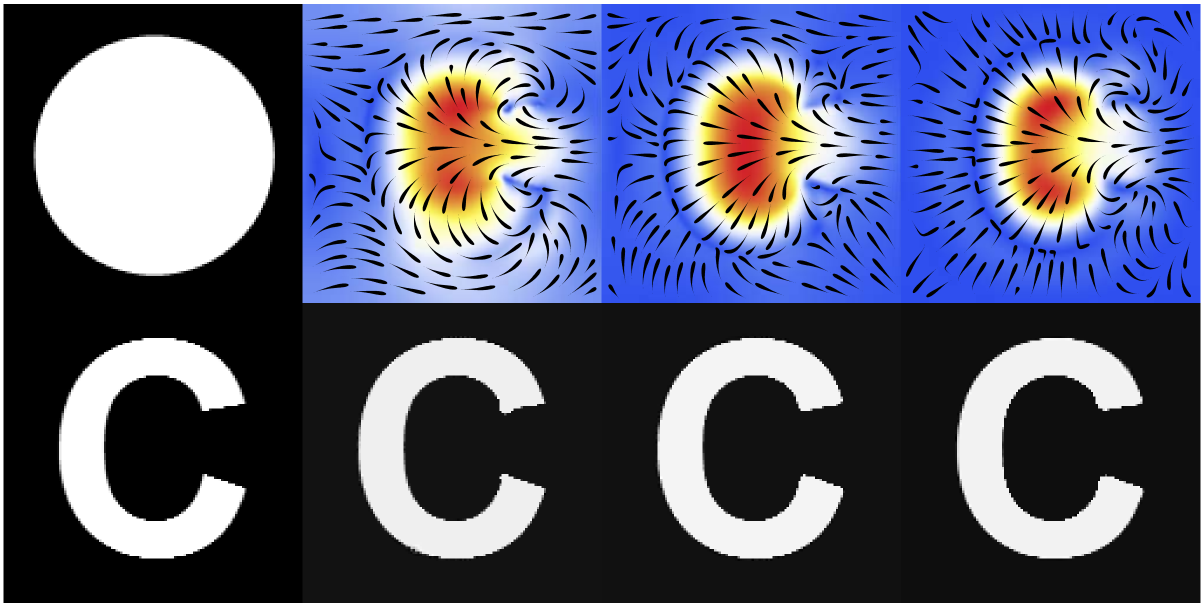

Fig. 1. The mapping of a circle (top left) onto the letter "C" (bottom left) with classical fluid fluid registration equation (1) (second column, $(\mu, \lambda) = (5/2, 0)$), the fluid diffusion equation (17) (third column, $\alpha = 1024$) and fluid optical flow equation (17) (last column $\alpha = 1024$)

tration, the smoothness constraint is defined as

$$\mathcal{T}_\mathrm{F}[\dot{\vec{u}}] = \frac{\rho}{2} \int_\Omega \sum_{i=1}^{2} \left(\Delta \dot{u}_i(\vec{x}, t)\right)^2 \, d^2\vec{x} \quad , \tag{3}$$

and for the diffusion registration

$$\mathcal{T}_\mathrm{D}[\dot{\vec{u}}] = -\frac{\rho}{2} \int_\Omega \sum_{i=1}^{2} \left(\nabla \dot{u}_i(\vec{x}, t)\right)^2 \, d^2\vec{x} \quad . \tag{4}$$

The negative sign in front of $\mathcal{T}_\mathrm{D}$ is chosen, because we wish a positive operator in the equations of motion applied to the acceleration field. The Euler-Lagrange equation for

$$\mathcal{L}(\dot{\vec{u}}, \vec{u}) = \mathcal{T}[\dot{\vec{u}}] - \mathcal{V}[\vec{u}] \tag{5}$$

yields

$$\rho \frac{d^2}{dt^2} \left(\Delta^2 \vec{u}\right) - \vec{F} = 0 \tag{6}$$

for the optical flow-based image registration and

$$\rho \frac{d^2}{dt^2} \left(-\Delta \vec{u}\right) - \vec{F} = 0 \tag{7}$$

for the diffusion registration, with

$$\vec{F} = -\left[T(\vec{x}) - S(\vec{x} - \vec{u}(\vec{x}, t))\right] \nabla S(\vec{x} - \vec{u}(\vec{x}, t)). \tag{8}$$

Both equations are free from dissipation of the energy $\mathcal{H} = \mathcal{T} + \mathcal{V}$ and the solution will start to oscillate and develop waves. Since we are interested in the minimum of $\mathcal{V}$ one has to add friction forces.

2.2. Dissipative Forces

Friction forces are not an integral part of the Euler-Lagrange equations but they can be included as additional summand in the force Eqns. (6) and (7). We will add two kinds of friction forces, one that acts independently of the position in space, and a second force that includes the spatial dependence by space derivatives of the velocity field. As position-independent friction force the function $\vec{f}_1 = -\gamma \dot{\vec{u}}$ will be used. The second dissipative force will depend on the partial derivatives of the velocity field. The Navier-Stokes equation would suggest a term proportional to $\vec{f}_\mathrm{v} = -\nu \Delta \dot{\vec{u}}$ (the viscosity term), and for the optical flow-based case a term $\vec{f}_\mathrm{v} = -\nu \Delta^2 \dot{\vec{u}}$ will be used.

Adding the two friction forces to the left hand side of the Eqns. (6) and (7), the fluid extension for the optical flow-based registration is given by the solution of the equation

$$\rho \frac{d^2}{dt^2} \left(\Delta^2 \vec{u}\right) - \vec{F} = -\gamma \, \dot{\vec{u}}(\vec{x}, t) - \nu \Delta^2 \dot{\vec{u}}(\vec{x}, t) \tag{9}$$

and for the diffusion-based version one gets

$$\rho \frac{d^2}{dt^2} \left(-\Delta \vec{u}\right) - \vec{F} = -\mu\gamma \, \dot{\vec{u}}(\vec{x}, t) - \mu\nu \Delta \dot{\vec{u}}(\vec{x}, t). \tag{10}$$

The only force free solution of the equations is obtained for the case that the displacement field $\lim_{t\to\infty} \vec{u}(\vec{x}, t)$ transforms the sample image onto the template image. In this case, the force $\vec{F}$ is zero and the friction will cause that the velocity

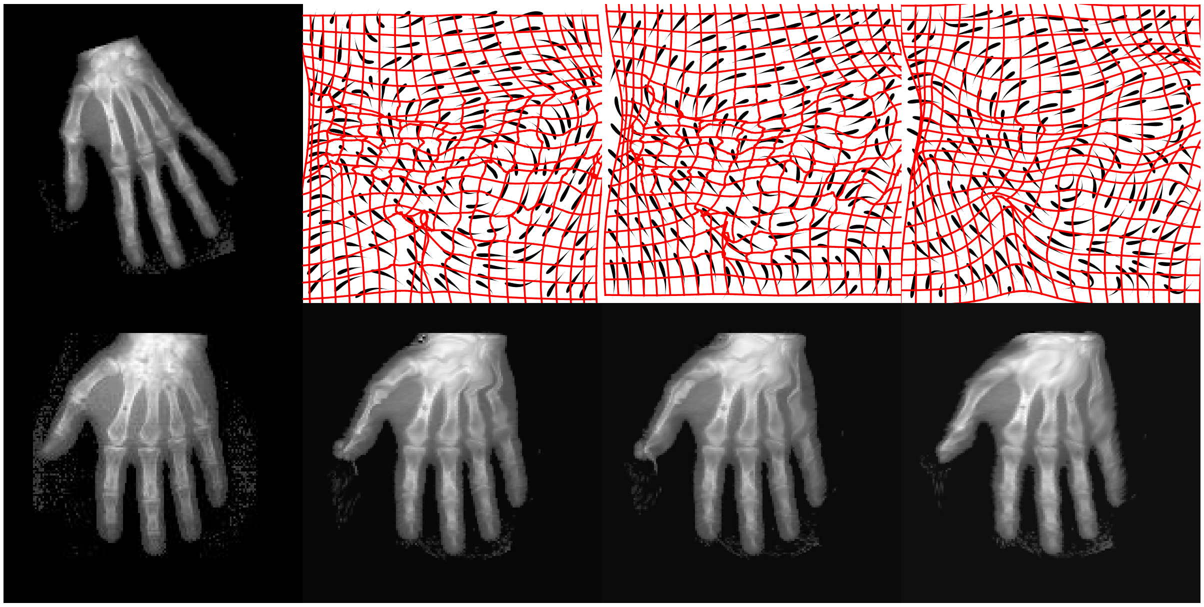

Fig. 2. The mapping of a x-ray image of a hand (top left) onto another hand (bottom left) with classical fluid registration equation (1) (second column, $(\mu, \lambda) = (3/2, 1/2)$), the fluid diffusion equation (17) (third column, $\alpha = 24576$) and fluid optical flow equation (17) (last column $\alpha = 3072$)

field goes also to zero. With the two Green functions

$$G_F[\Delta^2 g(\vec{x})] \quad = \Delta^2 G_F[g(\vec{x})] = \quad g(\vec{x}) \tag{11}$$

$$G_D[(-\Delta) g(\vec{x})] \quad = -\Delta G_D[g(\vec{x})] = \quad g(\vec{x}) \tag{12}$$

one gets the explicit system of second order equations that can be solved by a standard method for initial value problems. With either $G = G_F$ or $G = G_D$ the explicit version of the equations (9) and (10) can be written as

$$\rho\,\ddot{\vec{u}}(\vec{x}, t) = G\left[\vec{F}\right] - \gamma\, G\left[\dot{\vec{u}}(\vec{x}, t)\right] - \nu\,\dot{\vec{u}}(\vec{x}, t). \tag{13}$$

Since this are second order equations, the numerical solution is more expensive than the solution of the original fluid equation (1). For the image registration one is not interested in oscillations, and the overdamped limit of the equations can be taken by the assumption, that $\rho\,\ddot{\vec{u}}(\vec{x}, t)$ is much smaller than all other terms in the equation, so it can be neglected. Applying the differential operators of Eqns. (11) and (12) to the two variants of Eqn. (13) one gets two new equations for the image registration

$$\gamma\dot{\vec{u}}(\vec{x}, t) + \nu\,\Delta^2\dot{\vec{u}}(\vec{x}, t) = \vec{F} \tag{14}$$

for the optical flow-based registration and

$$\gamma\dot{\vec{u}}(\vec{x}, t) - \nu\,\Delta\dot{\vec{u}}(\vec{x}, t) = \vec{F} \tag{15}$$

the diffusion-based registration. It should be noted, that for $\gamma = 0$ one get the same formal result as for the fluid extension of the Navier-Lamé equation, i.e., replace formal the displacements with the velocities. It should also be noted, that the local damping force $\vec{f}_1$ removes the singularity of the differential operators and acts as a regularisation. To reduce the number of parameters in the method, it is convenient to absorb one in the scaled time variable $t \to t/\gamma$ and to introduce $\alpha = \nu/\gamma$. This gives the final form of the fluid extensionof the optical flow-based registration

$$\dot{\vec{u}}(\vec{x}, t) + \alpha\,\Delta^2\dot{\vec{u}}(\vec{x}, t) = \vec{F} \tag{16}$$

and for the diffusion-based registration

$$\dot{\vec{u}}(\vec{x}, t) - \alpha\,\Delta\dot{\vec{u}}(\vec{x}, t) = \vec{F}. \tag{17}$$

3. NUMERICAL SOLUTION

For the two discretely sampled images the Eqns. (16) and (17) will be solved on a regular grid. The partial derivatives in space can be approximated by finite differences. For the biharmonic operator in the optical flow-based equation the expression (25.3.33) from [8] is used, and the Laplace operator in the diffusion-based version is approximated by expression (25.3.31) [8]. Periodic boundary conditions for the computation of the explicit system of first order differential equations [9] are used. The initial value problem is solved using an explicit generalized Runge-Kutta-method [10] with error control and automatic step size adjustment.

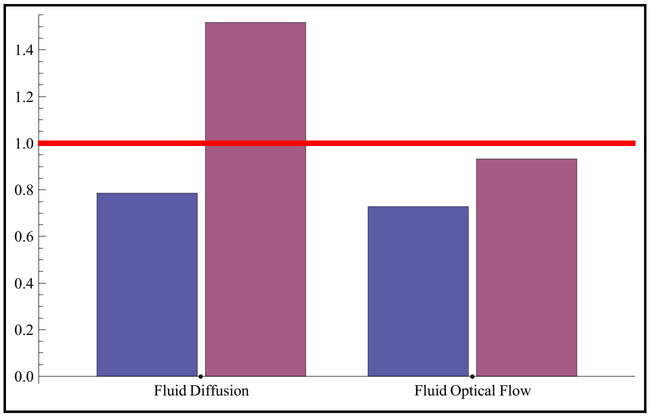

Fig. 3. Relative speed of the registration for the two examples (left circle to "C" mapping and right the registration of the hands) with the new equations to the execution time of the fluid registration Eqn. (1)

4. EXAMPLES

The integration of the equations of motion is stopped when the registered image $S(\vec{x} - \vec{u}(\vec{x}))$ does not change more than 2×10^{-2} per pixel. The intensities of sample and template image are all normalized to the interval $[0, 1]$.

The first example maps a circle on a 128×128 grid onto the letter "C" (Figure 1). The results differ in those regions where both sample and template image are black, and further in the positions of singular points in the flow field. The vortex on the edges of the letter is developed with all methods because this vortex is needed for the correct image match.

As a second example we show the transformation of two X-ray images of hands (Figure 2). Also in this case the results are very similar. The overlayed displacement of the regular mesh shows the lower smoothness of the fluid and diffusion registration compared with the optical flow-based variant, because the optical flow require the smoothness higher derivatives.

The comparison of the absolute speed of the new methods is not useful, because the absolute speed of the solution depends on the implementation details solving Eqns. (16) and (17). Our medium precision variable step size initial value solver [10] should outperform any fixed time step integration. That is the reason why only results relative to the solution of equationname (1) are shown. The new equations show a similar execution time than the original fluid registration. The fluid diffusion registration is always 20% faster than the original fluid registration and has similar smoothness properties of the computed transformation. The optical flow-based version is a bit slower but has superior smoothness properties.

5. CONCLUSIONS

We have presented two new equations for nonlinear image registration based on the optical flow and the diffusion registration. Instead of the introduction of an artificial time, the new equations require a smooth velocity field. Based on the underlying variational approach, the design of special fluid-like registration equations is possible by addition of future constrains for the displacement or the velocity field.

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